Variation of Constitutive Model Formulation on Analytical Cake Filtration Models

by

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Abstract

Constitutive properties are required to model porosity, effective stress, porewater pressure, or permeability during cake filtration. By using an analytical cake filtration model based on Tiller (1975) with various constitutive property formulations, predictions of effective stress, permeability, and porosity distributions in a filter cake can be made. This analytical approach was only valid when spatially average values of porosity and pressure differential were constant over time. Several different constitutive property formulations were compared to data of effective stress and permeability as a function of porosity for kaolin clay in water. Also the spatial distribution of porosity in a filter cake for kaolin clay in water was modeled using different constitutive relationships.

Introduction

In order to model cake filtration processes, an understanding of constitutive relationships is necessary. These relationships include the relationship between cake porosity (or concentration of solids), ε, and effective stress, σ', and porosity and permeability (or specific resistance), k. Such relationships have been deduced empirically by Tiller and Horng (1983) and Willis et al. (1985) using compression-permeability cell data and a variety of functional relationships between ε and k and ε and σ'.

The purpose of this paper is to examine how the various functional relationships can be used in an analytical cake filtration model. The results were compared to profiles of porosity of kaolin suspension filtrations described in Wells (1990).

Two-Phase Flow Governing Equations

The set of equations governing the behavior of the solid and liquid phases during pressure filtration in one dimension (neglecting the effects of gravity and unsteady and convective acceleration) [see Willis (1983), Tosun and Willis (1989)] consisted of continuity and momentum equations for each phase. These equations after averaging over a control volume of a filter cake are:

Liquid Continuity: $\qquad \dfrac{\partial \varepsilon}{\partial t} = \dfrac{\partial(\varepsilon V_l)}{\partial z}$ $\qquad\qquad\qquad\qquad$ **(1)**

Solid Continuity: $\qquad \dfrac{\partial \varepsilon}{\partial t} = \dfrac{-\partial((1-\varepsilon)V_s)}{\partial z}$ $\qquad\qquad\qquad$ **(2)**

Liquid Momentum: $\qquad (V_l - V_s) = \dfrac{k}{\varepsilon \mu}\dfrac{dp}{dz}$ $\qquad\qquad\qquad\qquad$ **(3)**

Solid Momentum: $\qquad (V_s - V_l) = \dfrac{k}{\varepsilon^2 \mu}\dfrac{d\sigma'}{dz} + \dfrac{(1-\varepsilon)k}{\varepsilon^2 \mu}\dfrac{dp}{dz}$ $\qquad$ **(4)**

where
- ε: porosity [-]
- t: time [T]
- z: distance from filter medium [L]
- V_l: liquid velocity [LT^{-1}]
- V_s: solid velocity [LT^{-1}]
- k: permeability [L^2]
- μ: dynamic viscosity [$ML^{-1}T^{-1}$]
- p: porewater pressure [$ML^{-1}T^{-2}$]
- σ': effective stress [$ML^{-1}T^{-2}$].

To solve for the six unknowns (ε, V_l, V_s, , k, and p) as functions of z and t, proper boundary and initial conditions and empirically deduced constitutive relationships were required.

By neglecting the medium pressure drop and assuming the solid velocities are zero, Tiller and Leu (1980) deduced the following relationship (see also Wells, 1999) for fractional cake position where L is the cake length and Δp is the applied pressure:

$$\frac{z}{L} = \frac{\int_{\sigma'}^{\Delta p} k \, d\sigma'}{\int_0^{\Delta p} k \, d\sigma'} \; . \qquad\qquad\qquad \textbf{(5)}$$

If the relationship between permeability and effective stress is known, an equation for the porosity, effective stress, and permeability distributions as a function of z/L can be determined (see Willis et al. 1991, Wells, 1999).

Constitutive Relationships

Relationships between porosity and permeability (or specific resistance) and between porosity and effective stress were necessary to close the above set of governing equations. Table 1 shows some of these constitutive relationships.

Table 1. Typical constitutive property relationships used in cake filtration.

#	Reference	Porosity-Effective Stress Relationship	Porosity-Specific Resistance or Permeability Relationship	Description
1	Wells (1990), Wells and Dick (1993)	$m_v = -\dfrac{\partial e}{\partial s'} = c \exp(de)$	$k = a \exp(be)$	where $a[L^2]$, $b[-]$, $c[M^{-1}LT^2]$, and $d[-]$ are empirical coefficients and m_v is the coefficient of volume compressibility $[M^{-1}LT^2]$
2	Shirato et al. (1970)	$(1-e) = (1-e_o)\left(1+\dfrac{s}{P_o}\right)^b$ or in terms of m_v $m_v = -\dfrac{\partial e}{\partial s'} =$ $(1-e_o)\dfrac{b}{P_o}\left[\dfrac{(1-e)}{(1-e_o)}\right]^{\frac{b-1}{b}}$	$a = a_o\left(1+\dfrac{s}{P_o}\right)^n$ or in terms of k* $k = k_o\left(\dfrac{(1-e)}{(1-e_o)}\right)^{-\frac{n}{b}-1}$	β [-] and n [-] are empirical parameters, ε_o [-] is a base porosity at which P_o is the effective stress at ε_o, α_o is the specific resistance associated with P_o, k_o is the permeability $[L^2]$ associated with ε_o, for a $CaCO_3$-H_2O system, ε_o=0.7766, P_o=85.5 kN/m^2, β=0.105, α_o= 4.79E10 m/kg, n=0.465 (Tan and Teoh, 1999)
3	Vorobjov et al. (1993)	$G = -(1+e)\dfrac{\partial s'}{\partial e} = G_o\left(\dfrac{s'}{P_o}\right)^n$ *or* $m_v = \dfrac{e^2}{(1-e)}\dfrac{\left(\dfrac{P_o}{s'}\right)^n}{G_o}$	$r = r_o\left(\dfrac{s'}{P_o}\right)^s$ or in terms of k and ε $k = k_o\left(\dfrac{1-e_o}{1-e}\right)^{\left(\frac{G_oS}{P_o}+1\right)}$ if r is interpreted as specific resistance or $k = k_o\left(\dfrac{1-e_o}{1-e}\right)^{\frac{G_oS}{P_o}}$ if r is merely the inverse permeability; this is similar to Coakley and Jones (1956) $a = a_o\left(\dfrac{\Delta p_c}{P_o}\right)^s$	G is the cake compressibility modulus [kPa], e is the void ratio $=\dfrac{e}{1-e}$, r is the resistance $[m^2]$, r_o is an empirical coefficient $[m^2]$, G_o is an empirical coefficient [kPa] and s is the compressibility coefficient [-]; typical values of coefficients were n=1, G_o=6900 kPa, s=0.5, r_o=1E14 m^2, Δp_c is the pressure differential across the cake (the final effective stress) αo is the cake resistance $[LM^{-1}]$ at P_o, an arbitrary pressure differential $[ML^{-1}T^{-2}]$, s is the coefficient of compressibility [-] with typical values between 0.5 and 1.1 (Dick and Ball, 1980)

* The relationship between permeability and effective stress is $a = \dfrac{1}{(1-e)\,r_s k}$ where ρ_s is the solid density

Determination of Constitutive Parameter Relationships

In order to determine the empirical coefficients necessary for the proposed constitutive relationships shown in Table 1, experimental data, such as relationships between cake porosity and permeability and porosity and effective stress, were necessary. Researchers have determined these parameters required by the constitutive property formulation by several techniques:

- Measurement of dynamic porewater pressure and porosity throughout the cake (Bierck et al., 1988, Wells and Dick, 1993)
- Use of compression-permeability cell (Tiller and Horng, 1983, and Willis et al., 1985, even though criticized by many as in Stamatakis and Tien, 1991)
- Use of specific resistance tests (Wells, 1990)
- Cake thickness and applied pressure over time for constant-rate filtration experiments (Stamatakis and Tien, 1991)

Determination of the empirical parameters is based on a statistical least-squares error approach where the parameters are chosen to minimize the sum of squares of the error.

An example data set for the filtration of kaolin clay in distilled water was used from Wells (1990). Figures 1 and 2 show the k-ε and σ'-ε relationships, respectively, using least-squares fits from Equation 1 and 2 from Table 1.

Analytical Models of Porosity, Effective Stress and Permeability

Analytical relationships for porosity, effective stress, and permeability as functions of fractional cake position, z/L, can be derived based on the assumed form of the constitutive relationship. This has been shown by Stamatakis and Tien (1991) and Wells (1999) and summarized by Willis et al. (1991).

Relationships are shown in Table 2, 3 and 4 for effective stress, permeability, and porosity, respectively, for Equation 1 and 2 from Table 1.

Table 2. Effective stress as a f(z/L) for constitutive relationships in Table 1.

Constitutive relationship #	Effective stress as a f(z/L)
1	$\sigma' = \Delta p - (cd)^{-1}\exp(-d\,\varepsilon_o) + (cd)^{-1}\left\{\left(1-\dfrac{z}{L}\right)\dfrac{z}{L}\exp[(b-d)\varepsilon_i]\right\}^{-\frac{d}{b-d}}$ where ε_o is the terminal porosity at z/L=0; ε_i is the initial porosity; Δp is the applied pressure differential; a, b, c, d are empirical parameters defined in Table 1
2	$\sigma' = P_o\left[\left\{\left(1-\left(1+\dfrac{\Delta P}{P_o}\right)^{a+1}\right)\left(\dfrac{z}{L}\right)+\left(1+\dfrac{\Delta P}{P_o}\right)^{a+1}\right\}^{\frac{1}{a+1}}-1\right]$ where $a=-n-b$, which are empirical parameters described in Table 1; P_o is a reference effective stress

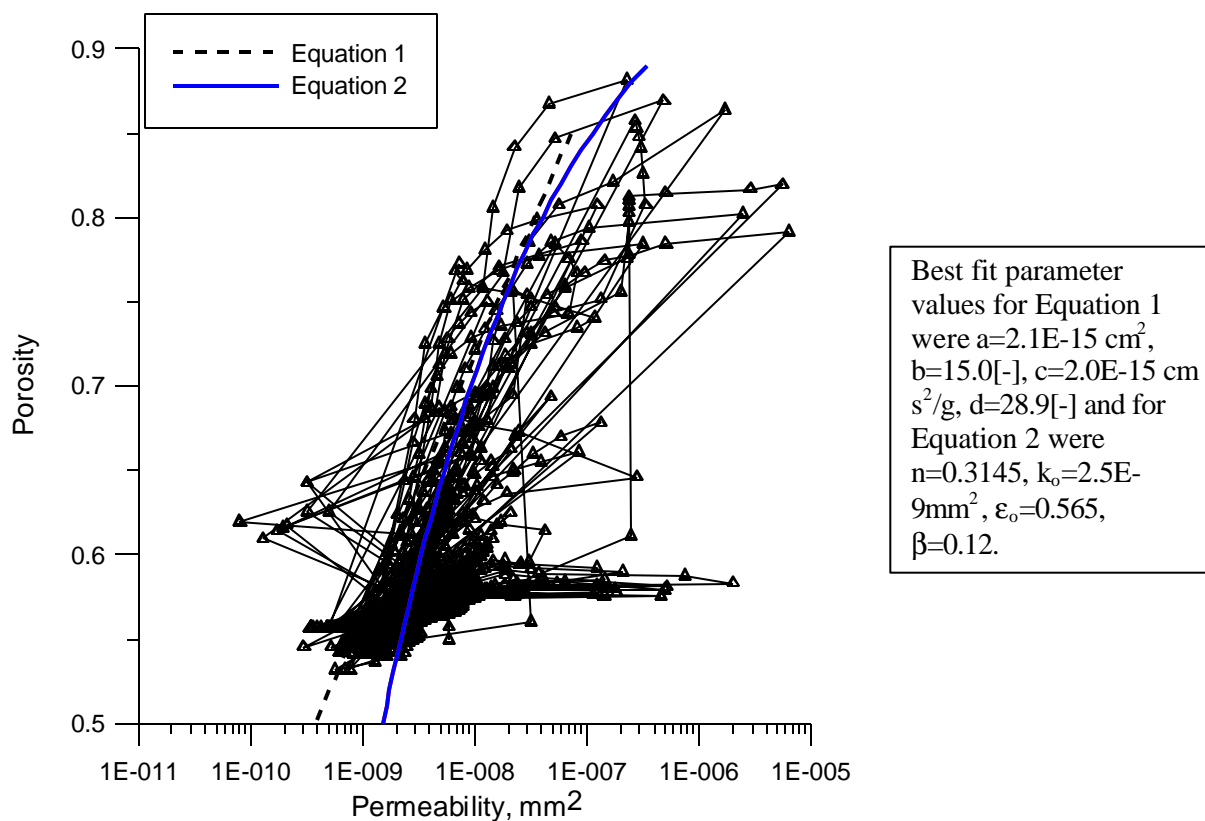Best fit parameter values for Equation 1 were a=2.1E-15 cm^2, b=15.0[-], c=2.0E-15 cm s^2/g, d=28.9[-] and for Equation 2 were n=0.3145, k_o=2.5E-9mm^2, ε_o=0.565, β=0.12.

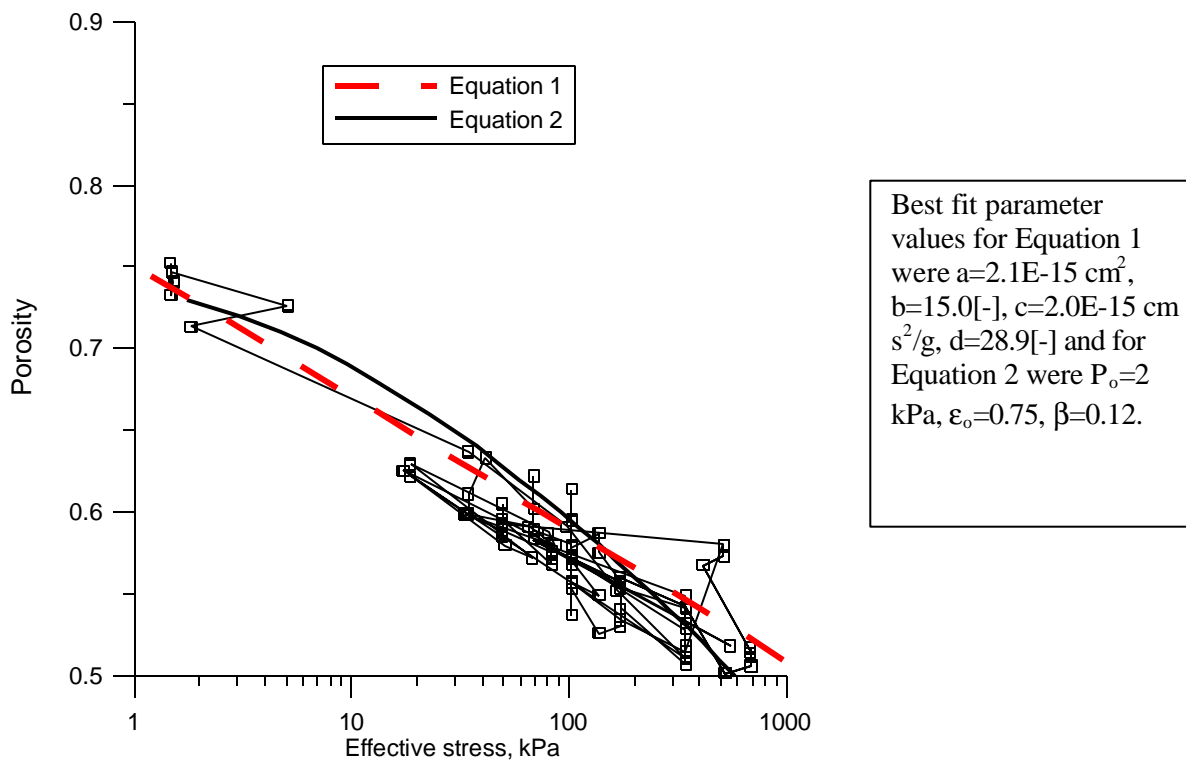Figure 1. Permeability-porosity relationship for kaolin suspensions (Wells, 1990) using Equations from Table 1.

Best fit parameter values for Equation 1 were a=2.1E-15 cm^2, b=15.0[-], c=2.0E-15 cm s^2/g, d=28.9[-] and for Equation 2 were P_o=2 kPa, ε_o=0.75, β=0.12.

Figure 2. Effective stress-porosity relationships for kaolin suspensions (Wells, 1990) using Equations from Table 1.

Table 3. Permebility as a f(z/L) for constitutive relationships in Table 1.

Constitutive relationship #	Permeability as a f(z/L)
1	$$k = a\left\{\left(1 - \frac{z}{L}\right)\exp[(b-d)e_o] + \frac{z}{L}\exp[(b-d)e_i]\right\}^{\frac{b}{b-d}}$$ where ε_o is the terminal porosity at z/L=0; ε_i is the initial porosity; Δp is the applied pressure differential
2	$$k = k_o\left[\left\{\left(1 - \left(1 + \frac{\Delta P}{P_o}\right)^{a+1}\right)\left(\frac{z}{L}\right) + \left(1 + \frac{\Delta P}{P_o}\right)^{a+1}\right\}^{\frac{1}{a+1}}\right]^a$$ where $a = -n - b$

Table 4. Porosity as a f(z/L) for constitutive relationships in Table 1.

Constitutive relationship #	Porosity as a f(z/L)
1	$$e = (b-d)^{-1}\ln\left\{(1 - \frac{z}{L})\exp((b-d)e_O) + \frac{z}{L}\exp((b-d)e_i)\right\}$$ where ε_o is the terminal porosity at z/L=0; ε_i is the initial porosity; Δp is the applied pressure differential
2	$$e = 1 - (1 - e_o)\left\{\left(1 - \left(1 + \frac{\Delta P}{P_o}\right)^{a+1}\right)\left(\frac{z}{L}\right) + \left(1 + \frac{\Delta P}{P_o}\right)^{a+1}\right\}^{\frac{b}{a+1}}$$ where $a = -n - b$

In order to show the functional dependence of these constitutive relationships, the relationships from Table 4 were compared to a set of dynamic profiles of porosity (or concentration of solids using $C = r_s[1 - e]$ where ρ_s is the density of the solids) of kaolin cake filtration experiments described in Wells (1990) in Figure 3.

Both analytical cake models with the given constitutive relationships predicted well the average profile of z/L vs. C, except during the initial period - effect of gravity sedimentation/filtration see Christiansen and Dick (1985) - and the final period - effect of surface tension, shrinkage - see Bierck et al. (1988). The scatter of the data about the theoretical curve was greater in the upper region of the cake where suspended solids concentrations were changing rapidly.

Whenever the average cake concentration was constant over time, the suspended solids concentration as a function of z/L was not a function of time and the assumptions in the model development were valid.
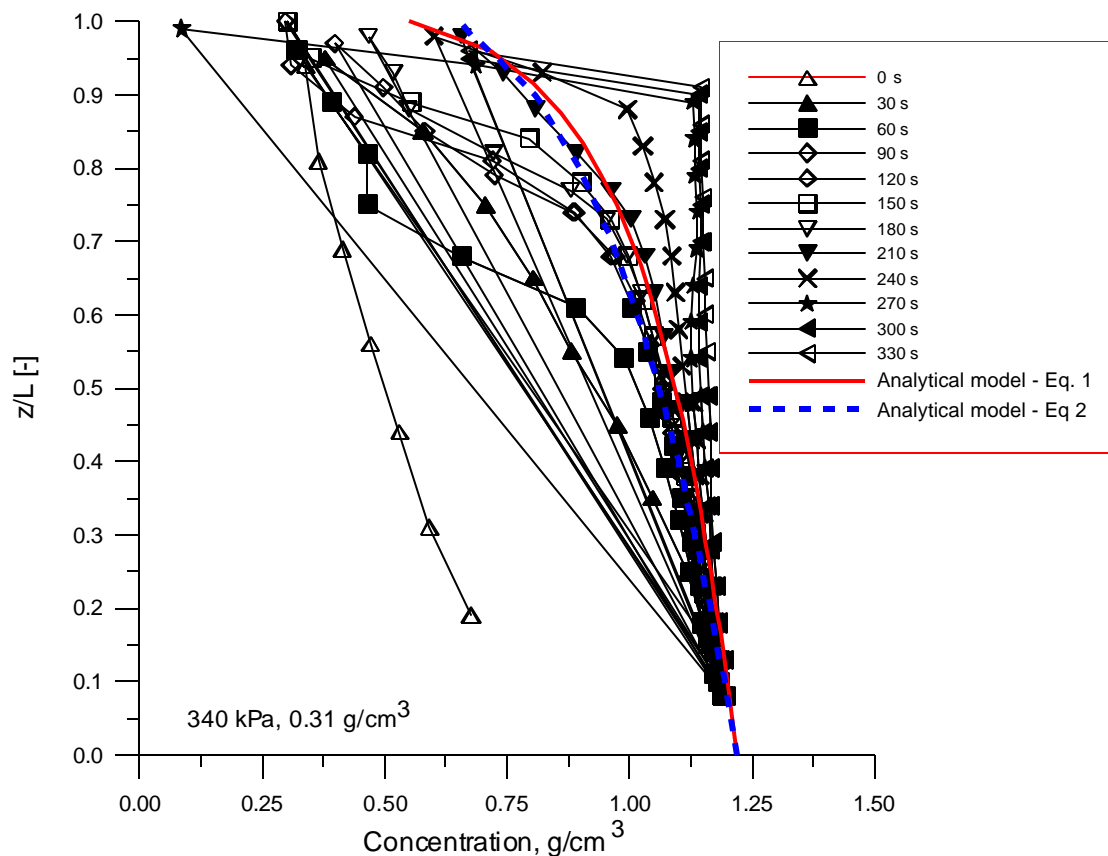
Figure 3. Concentration of solids as a function of fractional cake position for a kaolin suspension at 340 kPa pressure differential and an initial concentration of 0.31 g/cm^3.

Summary

Analytical models were developed that simulated the steady-state profiles of porosity, effective stress and permeability during cake filtration. The model assumptions were no solid velocities in the cake, which implied a constant average concentration of porosity or suspended solids concentration. The analytical model agreed well with porosity data only when the average cake concentration was constant (an assumption inherent in the development of the specific resistance test). During both the initial and final stages of dewatering the assumption of constant average cake porosity was not valid. Either functional form of the constitutive relationships used in the analytical model yielded reasonable model predictions compared to data.

The development of mathematical models of dewatering processes driven by laboratory derived constitutive relationships provides a tool for researchers to understand complicated dewatering phenomena and how conditions, such as a sludge's constitutive properties, initial suspension concentration distribution, pressure differential, and temperature, can affect filtrate production.

Even though a sludge's physical properties can be deduced from the specific resistance test (Wells, 1991), further research is required to assess constitutive relationships for chemical and biological sludges. Research is also needed to extend the application of mathematical dewatering models to mathematical models of full-scale dewatering processes.

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