Analytical Modeling of Cake Filtration

by

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Abstract

The specific resistance model of cake filtration is based on the assumption that the average cake resistance is constant over time. This implies also that the spatially average values of porosity and pressure differential are also constant over time since specific resistance is a function of both parameters.

An analytical cake filtration model based on Tiller (1975) was derived using constitutive properties of kaolin suspensions in distilled water. Constitutive properties were determined by experiments performed using high-energy X-rays and porewater pressure probes to determine the spatial and temporal distributions of porosity and porewater pressure. Using these constitutive properties, predictions were made of porewater pressure and porosity distributions using the analytical model. Porosity data were used to compare to the analytical model predictions assuming that porosity and pressure were functions only of fractional cake position.

Introduction

The specific resistance is used to characterize biosolids or sludge dewaterability (Christiansen and Dick, 1985a). Descriptions of the specific resistance test are shown in Suidan (1988) and Vesilind (1979). An assumption in the development of the specific resistance model is that the solid velocities are negligible throughout the cake. Work by Tiller et al. (1987) and Willis et al. (1985) showed that whenever the distribution of porosity was only a function of fractional cake position,

z/L, where z is the height above the medium and L is the cake length, this assumption of the specific resistance model was appropriate.

Deviations from the assumption of constant average porosity and porewater pressure were evaluated during this study. For many experiments, the average porosity and pressure differential across the cake during a Buchner funnel test is not constant during the initial and final stages of dewatering (Bierck et al., 1988).

Two-Phase Flow Governing Equations

The set of equations governing the behavior of the solid and liquid phases during pressure filtration in one dimension (neglecting the effects of gravity and unsteady and convective acceleration) [see Willis (1983), Tosun and Willis (1989)] consisted of continuity and momentum equations for each phase. These equations after averaging over the control volume for the domain of the cake shown in Fig. 1 are:

Liquid Continuity:
$$\frac{\partial e}{\partial t} = \frac{\partial(eV_l)}{\partial z} \tag{1}$$

Solid Continuity:
$$\frac{\partial e}{\partial t} = \frac{-\partial((1-e)V_s)}{\partial z} \tag{2}$$

Liquid Momentum:
$$(V_l - V_s) = \frac{k}{em}\frac{dp}{dz} \tag{3}$$

Solid Momentum:
$$(V_s - V_l) = \frac{k}{e^2 m}\frac{ds'}{dz} + \frac{(1-e)k}{e^2 m}\frac{dp}{dz} \tag{4}$$

where
ε:	porosity [-]
t:	time [T]
z:	distance from filter medium [L]
V_l:	liquid velocity [LT^{-1}]
V_s:	solid velocity [LT^{-1}]
k:	permeability [L^2]
μ:	dynamic viscosity [$ML^{-1}T^{-1}$]
p:	porewater pressure [$ML^{-1}T^{-2}$]
σ':	effective stress [$ML^{-1}T^{-2}$].

To solve for the six unknowns (ε, V_l, V_s, , k, and p) as functions of z and t, proper boundary and initial conditions and empirically deduced constitutive relationships were required.

Constitutive Relationships

Relationships between porosity and permeability (or specific resistance) and between porosity and effective stress were necessary to close the above set of governing equations. Such relationships were deduced empirically by Tiller and Horng (1983) and Willis et al. (1985) using compression-permeability cell data and a variety of functional relationships between ε and k and ε and σ'. Experimental work, summarized in Wells and Dick (1993) and Wells (1990), determined these constitutive properties by using high-resolution dynamic porosity and porewater pressure data

during filtration experiments using kaolin clay in distilled water. The following constitutive relationships were used in that study of kaolin filtration:

$$k = a \exp (be)$$ (5)

$$m_v = -\frac{\partial e}{\partial s'} = c \exp (de)$$ (6)

where $a[L^2]$, $b[-]$, $c[M^{-1}LT^2]$, and $d[-]$ are empirical coefficients and m_v is the coefficient of volume compressibility $[M^{-1}LT^2]$ [see Ortique (1982)]. For kaolin clay suspensions, these empirical coefficients were found to be a=2.1E-15 cm^2, b=15.0[-], c=2.0E-15 cm s^2/g, d=28.9[-].

Analytical Model Development and Results

Shirato and Sambuichi (1964) and Tiller (1975) developed an approach for analytical modeling of the porosity distribution within a filter cake assuming that solid velocities were negligible compared to liquid velocities within the cake. [Note that the assumption of zero solid velocities was equivalent to assuming that the average cake porosity was constant because Eq. 2 with $V_s = 0$ would require that ε could not be a f(t).]

Using the assumption that $V_s = 0$, analytical relationships for porosity, porewater pressure, effective stress, and permeability as functions of fractional cake position, z/L, were derived in Appendix A based on Tiller's (1975) approach and the constitutive relationships of Eqs. 5 and 6.

Using the constitutive relationship parameter values for kaolin clay, comparisons of the analytical model predictions (Eq.) and dynamic porosity profile data are shown in Figs. 1 to 4 for pressure differentials of 170 kPa, 340 kPa, 520 kPa and 690 kPa, respectively. The dynamic porosity data were plotted as suspended solids concentration vs the normalized distance coordinate, z/L, where L is the length of the cake and z is the distance from the filtration medium. (Note that the initial data points in Figs. 1 to 4 are at t=0 s, which was the suspended solids concentration profile prior to application of the applied pressure differential.)

The analytical cake model with the given constitutive relationships predicted the average profile of z/L vs. C, except during the initial period - effect of gravity sedimentation/filtration see Christiansen and Dick (1985b) - and the final period - effect of surface tension, shrinkage - see Bierck et al. (1988). The scatter of the data about the theoretical curve was greater in the upper region of the cake where suspended solids concentrations were changing rapidly.

Whenever the average cake concentration was constant over time, the suspended solids concentration as a function of z/L was not a function of time. For a constant average cake concentration, plots of t/V vs. V were linear, and the specific resistance concept was valid.

Summary

Analytical models were developed that simulated the steady-state profiles of porosity, porewater pressure, effective stress and permeability during cake filtration. The model assumptions were no solid velocities in the cake, which implied a constant average concentration of porosity or suspended solids concentration. The analytical model agreed well with porosity data only when the average cake concentration was constant (an assumption inherent in the development of the specific resistance test). During both the initial and final stages of dewatering the assumption of constant average cake porosity was not valid.
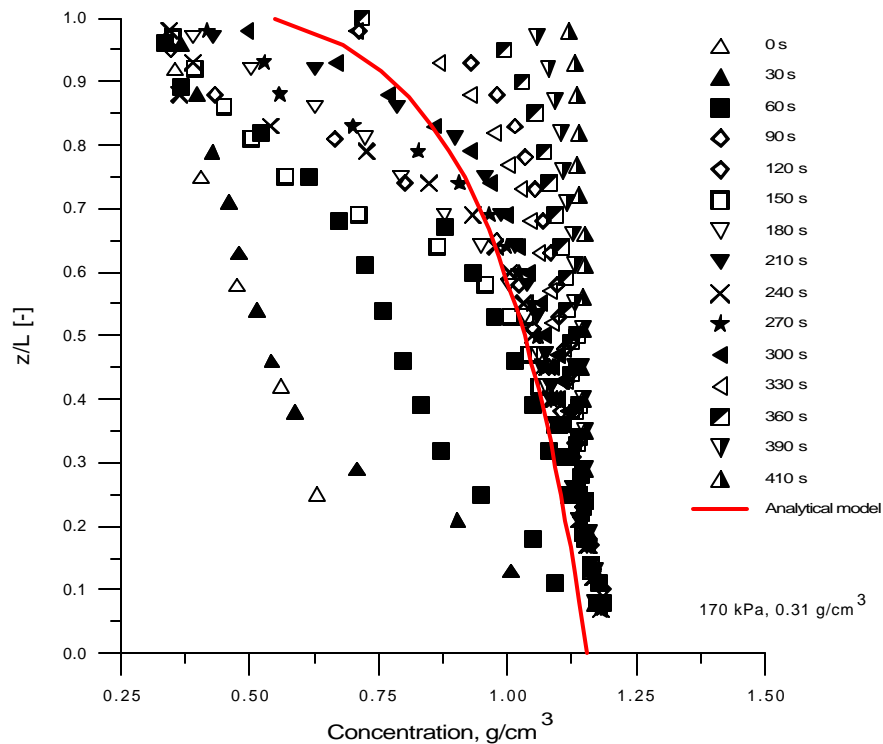
Figure 1. Concentration of kaolin suspended solids as a function of fractional cake position for pressure differential of 170 kPa.
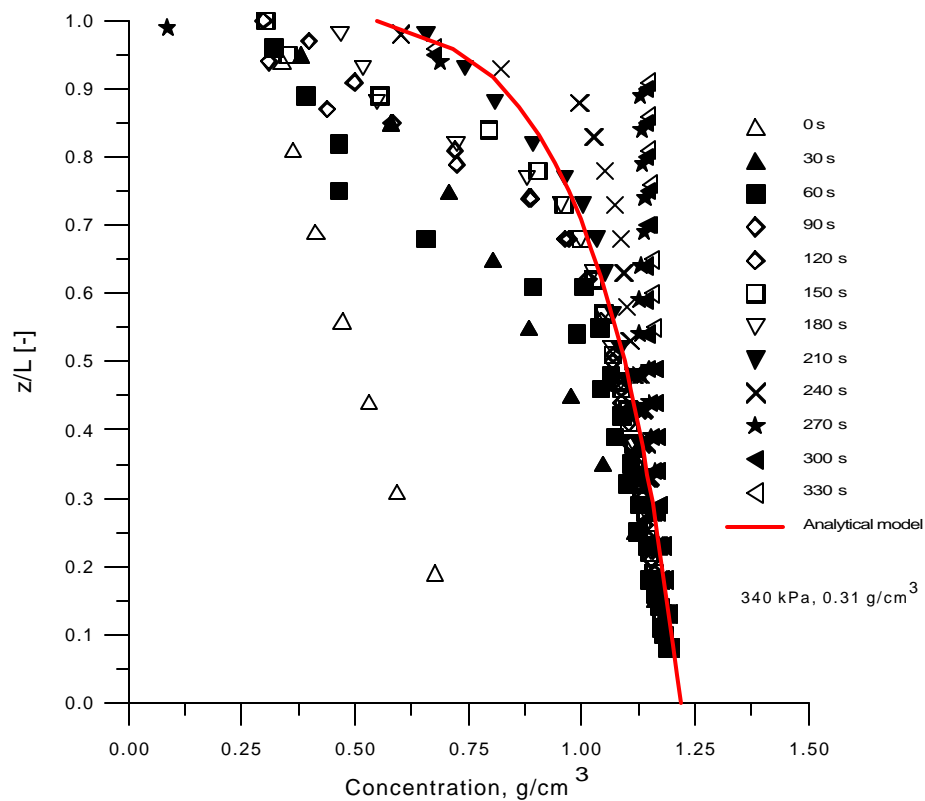

Figure 2. Concentration of kaolin suspended solids as a function of fractional cake position for pressure differential of 340 kPa.
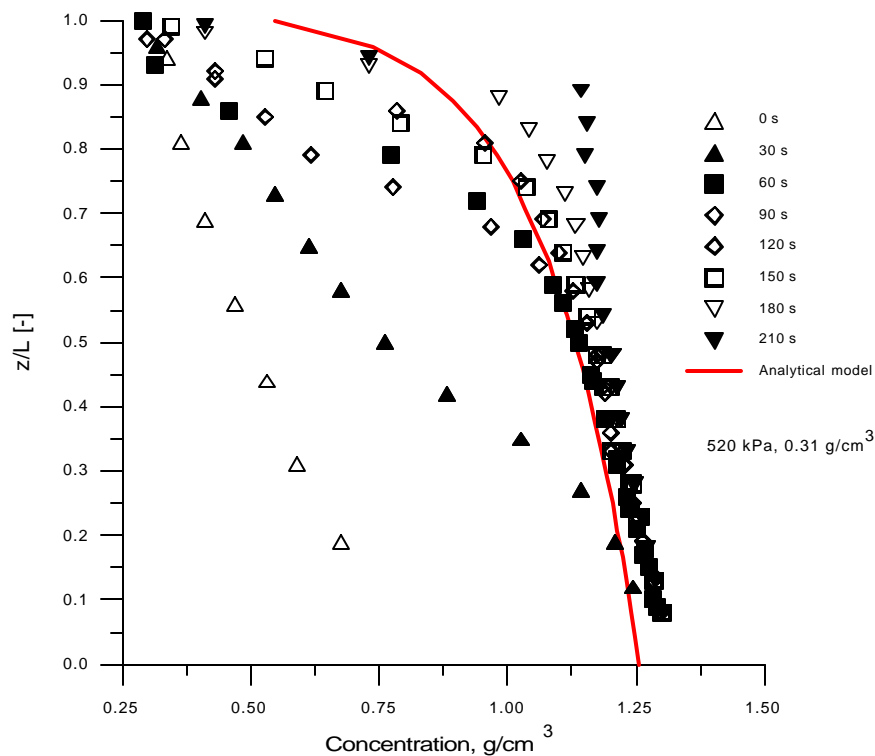
Figure 3. Concentration of kaolin suspended solids as a function of fractional cake position for pressure differential of 520 kPa.
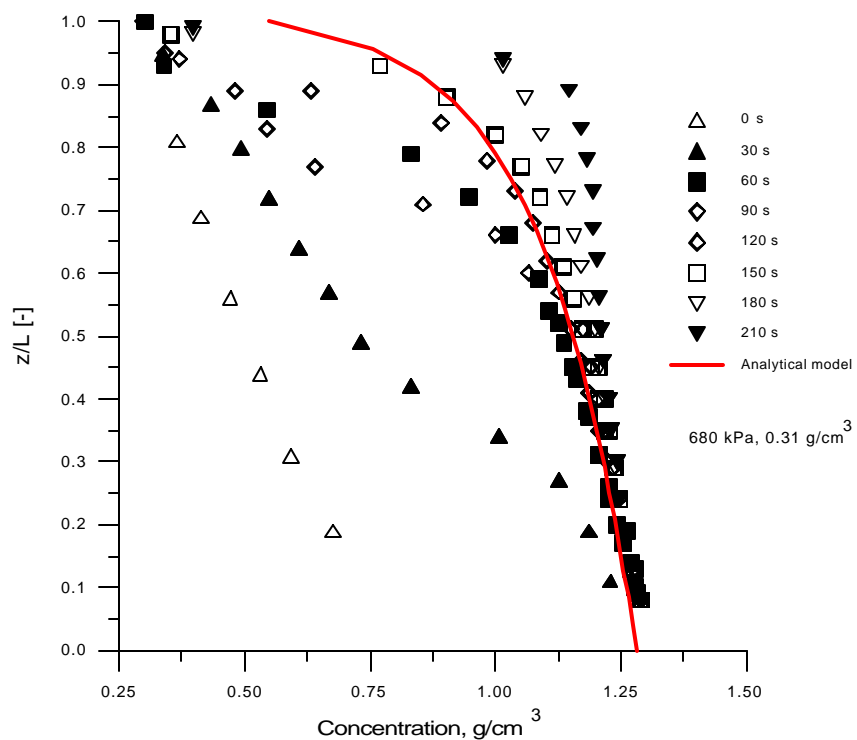

Figure 4. Concentration of kaolin suspended solids as a function of fractional cake position for pressure differential of 680 kPa.

The development of mathematical models of dewatering processes driven by laboratory derived constitutive relationships provides a tool for researchers to understand complicated dewatering phenomena and how conditions, such as a sludge's constitutive properties, initial suspension concentration distribution, pressure differential, and temperature, can affect filtrate production.

Even though a sludge's physical properties can be deduced from the specific resistance test (Wells, 1991), further research is required to assess constitutive relationships for chemical and biological sludges. Research is also needed to extend the application of mathematical dewatering models to mathematical models of full-scale dewatering processes.

Acknowledgments

References

Bierck, B.R., Wells, S.A., and Dick, R.I. (1988). "Compressible Cake Filtration: Monitoring Cake Formation and Shrinkage Using Synchrotron X-Rays," J. Water Pollution Control Fed., 60, 645-650.

Christiansen, G. and Dick, R. I. (1985a) "Specific Resistance Measurements: Methods and Procedures," J. of Envir. Engr., ASCE, 111, 258-271.

Christiansen, G. and Dick, R. I. (1985b) "Specific Resistance Measurements: Nonparabolic Data," J. of Envir. Engr., ASCE, 111, 243-257.

Ortique, A. (1982). "A Study of Physical Properties of Dewatered Municipal Sludges for Ultimate Disposal by Landfill," M.S. Thesis, School of Civ. and Envir. Engr., Cornell Univ., Ithaca, N.Y.

Shirato, M. and Sambuichi, M. (1964). "Experimental and Theoretical Studies in Constant Pressure Filtration of Thick Slurries," Chem. Engr. Japan, 2, 6-10.

Suidan M. T. (1988) "Environmental Engineering Unit Operations and Unit Processes Laboratory Manual," 4th edition, Association of Environmental Engineering Professors.

Tiller, F.M. (1975). "Compressible Cake Filtration," in Scientific Basis of Filtration, ed. by K. Ives, Noordhoff, London, 315-397.

Tiller, F.M. and Horng, L. (1983). "Hydraulic Deliquoring of Compressible Filter Cakes," AICHE J., 29, 297-305.

Tosun, I. (1986). "Formulation of Cake Filtration," Chem. Engr. Science, 41, 2563-2568.

Tosun, I. and Willis, M.S. (1989). "Making the Case for the Multiphase Filtration Theory," Filtration and Separation, July/August, 295-299.

Vesilind, P. A. (1979) Treatment and Disposal of Wastewater Sludges, Ann Arbor Science, Ann Arbor, MI.

Wells, S.A. (1990). "Compressible Cake Modeling and Analysis," Ph.D. Dissertation, School of Civ. and Envir. Engr., Cornell Univ., Ithaca, N.Y.

Wells, S. A. (1991) "Determination of Sludge Properties for Modeling Compressible Cake Filtration from Specific Resistance Tests," Proceedings ASCE National Conference in Environmental Engineering, ed. by C. O'Melia, Arlington, Va., 125-131.

Wells, S. A. and Dick, R. I. (1993) ""Permeability, Solid and Liquid Velocity, and Effective Stress Variations in Compressible Cake Filtration," in Advances in Filtration and Separation, Vol. 7, ed. by W. Leung, American Filtration and Separations Society, pp. 9-12.

Willis, M.S. (1983). "A Multiphase Theory of Filtration," Progress in Filtration and Separation 3, ed. R.J. Wakeman, Elsevier, Amsterdam.

Willis, M.S.; Tosun I.; and Collins, R.M. (1985). "Filtration Mechanisms," Chem. Engr. Res. Design, 63, 175-183.

Appendix - Analytical Cake Filtration Model Derivation

This appendix presents an analytical model of porosity, permeability, effective stress, and porewater pressure distributions in filter cakes.

Eq. 3 and 4 can be combined into a total momentum balance such that

$$\frac{dp}{dz} + \frac{d\mathbf{s}'}{dz} = 0 \quad or \quad \frac{dp}{dz} = -\frac{d\mathbf{s}'}{dz} \tag{7}$$

Substituting Eq. 7 into Eq. 3,

$$\frac{-k}{\mathbf{em}} \frac{\partial \mathbf{s}'}{\partial z} = (V_l - V_s) . \tag{8}$$

Separating the variables and integrating between the limits z_1 and z_2, Eq. 8 becomes

$$\int_{z1}^{z2} dz = \int_{\mathbf{s}'(z1)}^{\mathbf{s}'(z2)} \frac{-k\, d\mathbf{s}'}{\mathbf{m}(V_l - V_s)} . \tag{9}$$

Neglecting the medium pressure drop, $\sigma'(z=0)=p$ and $\sigma'(z=L)=0$. Integrating Eq. 9 between $z_1=0$ and $z_2=z$,

$$z = \int_{\Delta p}^{\mathbf{s}'} \frac{-k\, d\mathbf{s}'}{\mathbf{e}(V_l - V_s)\mathbf{m}} . \tag{10}$$

Similarly integrating Eq. 9 over the cake from $z_1=0$ to $z_2=L$,

$$L = \int_{\Delta p}^{0} \frac{-k\, d\mathbf{s}'}{\mathbf{e}(V_l - V_s)} . \tag{11}$$

Dividing Eq. 10 by Eq. 11 yields

$$\frac{z}{L} = \frac{\int_{\Delta p}^{\mathbf{s}'} \dfrac{k\, d\mathbf{s}'}{\mathbf{e}(V_l - V_s)}}{\int_{\Delta p}^{0} \dfrac{k\, d\mathbf{s}'}{\mathbf{e}(V_l - V_s)}} \tag{12}$$

Assuming $V_s=0$, reversing the limits of integration, and simplifying, Eq. 12 becomes

$$\frac{z}{L} = \frac{\int_{\mathbf{s}'}^{\Delta p} k\, d\mathbf{s}'}{\int_{0}^{\Delta p} k\, d\mathbf{s}'} . \tag{13}$$

If the relationship between permeability and effective stress is known, an equation for the porosity distribution as a function of z/L can be determined [Tiller and Leu (1980)]. Alternatively, using the definition of m_v in Eq. 6, i.e.,

$$d\boldsymbol{s}' = \frac{\partial \boldsymbol{s}'}{\partial \boldsymbol{e}} d\boldsymbol{e} = \frac{-d\boldsymbol{e}}{m_v} \; , \tag{14}$$

with appropriate boundary conditions (at the top of the cake at z=L the porosity is ε_i, the initial porosity of the suspension, and at the bottom of the cake at z=0, $\varepsilon=\varepsilon_o$, the terminal porosity) and substituting these into Eq. 13,

$$\frac{z}{L} = \frac{\int_{\boldsymbol{e}}^{\boldsymbol{e}_o} \frac{k}{m_v} d\boldsymbol{e}}{\int_{\boldsymbol{e}_i}^{\boldsymbol{e}_o} \frac{k}{m_v} d\boldsymbol{e}} \; . \tag{15}$$

Substituting Eqs. 5 and 6 into Eq. 15, simplifying, and rearranging, then

$$\boldsymbol{e}(\frac{z}{L}) = (b-d)^{-1} \ln\{(1 - \frac{z}{L}) \exp((b-d)\boldsymbol{e}_o) + \frac{z}{L} \exp((b-d)\boldsymbol{e}_i)\} \; . \tag{16}$$

Porewater pressure, effective stress, and permeability profiles also can be determined as functions of position, z/L. The permeability as a function of z/L is found by substituting Eq. 16 into Eq. 5 and simplifying such that

$$k(\frac{z}{L}) = a\{ (1 - \frac{z}{L}) \exp[(b-d)\boldsymbol{e}_o] + \frac{z}{L} \exp[(b-d)\boldsymbol{e}_i] \}^{\frac{b}{b-d}} \; . \tag{17}$$

By integrating Eq. 6, i.e., $\quad \int_{\boldsymbol{e}_o}^{\boldsymbol{e}} \frac{d\boldsymbol{e}}{c \exp(d\boldsymbol{e})} = \int_{\Delta p}^{\boldsymbol{s}'} \cdot d\boldsymbol{s}' \; , \tag{18}$

then

$$\boldsymbol{s}'(\frac{z}{L}) = \Delta p + (cd)^{-1} \left(\exp(-d\boldsymbol{e}) - \exp(-d\boldsymbol{e}_o) \right) \; . \tag{19}$$

Substituting Eq. 16 into Eq. 19 yields

$$\boldsymbol{s}'(\frac{z}{L}) = \Delta p - (cd)^{-1} \exp(-d\boldsymbol{e}_o) + (cd)^{-1} \left\{ \left(1 - \frac{z}{L} \right) \right.$$

$$\left. \frac{z}{L} \exp[(b-d)\boldsymbol{e}_i] \right\}^{-\frac{d}{b-d}} \; . \tag{21}$$

Using Eq. 7 and appropriate boundary conditions with Eq. 21, the porewater pressure is

$$p(\frac{z}{L}) = (cd)^{-1} \exp(-d\boldsymbol{e}_o) - (cd)^{-1} \left\{ \left(1 - \frac{z}{L} \right) \right.$$

$$\left. \frac{z}{L} \exp[(b-d)\boldsymbol{e}_i] \right\}^{-\frac{d}{b-d}} \; . \tag{22}$$